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A Metric Space Connected with Generalized Means[1]

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1. INTRODUCTION

DEFINITION 1.1. Let $-\infty < a < b < \infty$. Then $\mathrm{CM}[a,b]$ denotes the set of all functions with domain $[a,b]$ that are continuous and strictly monotonic there.

DEFINITION 1.2. Let $-\infty < a < b < \infty$, and let $f \in \mathrm{CM}[a,b]$. Then, for each positive integer n, each n-tuple $x = (x_1, x_2, \ldots, x_n)$ where $a \leqslant x_j \leqslant b$ ($j = 1, 2, \ldots, n$), and each n-tuple $q = (q_1, q_2, \ldots, q_n)$ where $q_j > 0$ ($j = 1, 2, \ldots, n$) and $\sum_{j=1}^{n} q_j = 1$, let $M_f(x,q)$ denote the (weighted) mean $f^{-1}\{\sum_{j=1}^{n} q_j f(x_j)\}$. For the sake of brevity, we say that x and q are *admissible* if, for some positive integer n, they are n-tuples satisfying the conditions specified above.

Remarks. (1) For a detailed treatment of these generalized means, see [*1*], Ch. III. (2) Clearly, $a \leqslant M_f(x,q) \leqslant b$ holds for all admissible x and q. (3) If $0 < a < b$, $q_1 = q_2 = \ldots = q_n = 1/n$, and $f(t)$ is t^{-1}, $\log t$, or t, then $M_f(x,q)$ is the ordinary harmonic mean, geometric mean, or arithmetic mean, respectively. (4) If f, $g \in \mathrm{CM}[a,b]$ and if g is increasing, then $M_f(x,q) \leqslant M_g(x,q)$ for all admissible x and q if and only if the composite function $g \circ f^{-1}$ is convex on $[f(a), f(b)]$ (or $[f(b), f(a)]$). (Cf. [*1*], p. 75.)

Suppose that one is given an $f \in \mathrm{CM}[a,b]$ and that one has to compute $M_f(x,q)$ for various admissible x and q. Since rounding-off errors may be involved in such computations, one is prompted to ask if, given an $\epsilon > 0$, there exists a polynomial with rational coefficients, whose restriction p to $[a,b]$ belongs to $\mathrm{CM}[a,b]$, such that $|M_f(x,q) - M_p(x,q)| < \epsilon$ for all admissible x and q. Theorem 2.2 of this paper asserts that such a polynomial exists. It turns out that Theorem 2.2 also proves that a certain metric space is separable.

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[1] A lecture delivered at the Second Symposium on Inequalities at USAF Academy, Colorado, August 1967.

(Cf. Corollary 2.1.) The main purpose of this paper is to introduce this metric space and to analyze it in detail.

2. Separability of the Metric Space of Equivalence Classes

First, we observe that, for the purpose of forming means, certain functions in $\mathrm{CM}[a,b]$ are equivalent.

Lemma 2.1 ([1], p. 66). *If $F, f \in \mathrm{CM}[a,b]$, then $M_F(x,q) = M_f(x,q)$ for all admissible x and q if and only if there exist real numbers α ($\neq 0$) and β such that $F = \alpha f + \beta$.*

Definition 2.1. We say that functions $F, f \in \mathrm{CM}[a,b]$ are equivalent, and we write $F \sim f$, if and only if there exist real numbers α ($\neq 0$) and β such that $F = \alpha f + \beta$.

We have the following obvious

Lemma 2.2. *$\sim$ is an equivalence relation on* $\mathrm{CM}[a,b]$.

Definition 2.2. If $f \in \mathrm{CM}[a,b]$, then $[f]$ denotes the equivalence class containing f.

In view of the approximation problem propounded in Section 1, and in view of Lemma 2.1, we make the following

Definition 2.3. If $f, g \in \mathrm{CM}[a,b]$, we define the distance $\rho([f],[g])$ between $[f]$ and $[g]$ by $\rho([f],[g]) = \sup\{|M_f(x,q) - M_g(x,q)| : x \text{ and } q \text{ admissible}\}$.

Remark. The supremum in Definition 2.3 is actually attained; but we defer the proof until the end of the paper.

Theorem 2.1. *ρ is a well-defined metric on the set of all equivalence classes of* $\mathrm{CM}[a,b]$.

Proof. Observe that $|M_f(x,q) - M_g(x,q)| \leqslant b - a$ for all admissible x and q, and, therefore, $0 \leqslant \rho([f],[g]) \leqslant b - a$.

ρ is well defined since, according to Lemma 2.1 and Definition 2.1,

$$|M_F(x,q) - M_G(x,q)| = |M_f(x,q) - M_g(x,q)|$$

for all admissible x and q if $F \sim f$ and $G \sim g$.

Clearly, $\rho([f],[g]) = 0$ if and only if $[f] = [g]$. Finally, ρ is obviously symmetric; and, as is easily seen, it satisfies the triangle inequality.

Next, we prove that the metric space in question is separable. It is convenient to confine our attention to the unit interval $[0,1]$. We start with the following lemma, whose proof is evident.

LEMMA 2.3. *Given* $f \in \mathrm{CM}[0,1]$, *there exists precisely one function* $F \in \mathrm{CM}[0,1]$ *such that* $F \sim f$, $F(0)=0$, *and* $F(1)=1$.

DEFINITION 2.4. If $F \in \mathrm{CM}[0,1]$, $F(0)=0$, and $F(1)=1$, then we say that F is canonical.

According to Lemma 2.3, each equivalence class of $\mathrm{CM}[0,1]$ contains precisely one canonical function.

THEOREM 2.2. *Suppose that* $f \in \mathrm{CM}[0,1]$. *Then, corresponding to each* $\epsilon > 0$, *there exists a canonical polynomial*[2] p *with rational coefficients such that* $\rho([f],[p]) \leqslant \epsilon$.

Proof. We can assume without loss of generality that f is canonical. Then f^{-1} is also canonical.

If $\epsilon > 0$, then, by the uniform continuity of f^{-1}, there exists a $\delta > 0$ such that $|f^{-1}(y_1) - f^{-1}(y_2)| < \frac{1}{2}\epsilon$ if y_1, y_2 are in $[0,1]$ and $|y_1 - y_2| < \delta$.

Suppose that p is a canonical function such that $|f(t) - p(t)| < \delta$ throughout $[0,1]$. Then, as we now prove, $\rho([f],[p]) \leqslant \epsilon$. Let n be a positive integer, and let $x = (x_1, x_2, \ldots, x_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be admissible. Then

$$\begin{aligned}
|M_f(x,q) - M_p(x,q)| &= \left| f^{-1}\left\{\sum_{j=1}^{n} q_j f(x_j)\right\} - p^{-1}\left\{\sum_{j=1}^{n} q_j p(x_j)\right\}\right| \\
&\leqslant \left| f^{-1}\left\{\sum_{j=1}^{n} q_j f(x_j)\right\} - f^{-1}\left\{\sum_{j=1}^{n} q_j p(x_j)\right\}\right| \\
&\quad + \left| f^{-1}\left\{\sum_{j=1}^{n} q_j p(x_j)\right\} - p^{-1}\left\{\sum_{j=1}^{n} q_j p(x_j)\right\}\right| \\
&= \left| f^{-1}\left\{\sum_{j=1}^{n} q_j f(x_j)\right\} - f^{-1}\left\{\sum_{j=1}^{n} q_j p(x_j)\right\}\right| \\
&\quad + \left| f^{-1}\left\{p\left[p^{-1}\left(\sum_{j=1}^{n} q_j p(x_j)\right)\right]\right\} - f^{-1}\left\{f\left[p^{-1}\left(\sum_{j=1}^{n} q_j p(x_j)\right)\right]\right\}\right| \\
&< \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon,
\end{aligned}$$

[2] I.e., a restriction of a polynomial to [0, 1] that is canonical.

since

$$\left|\sum_{j=1}^{n} q_j f(x_j) - \sum_{j=1}^{n} q_j p(x_j)\right| = \left|\sum_{j=1}^{n} q_j\{f(x_j) - p(x_j)\}\right|$$

$$\leqslant \sum_{j=1}^{n} q_j |f(x_j) - p(x_j)|$$

$$< \sum_{j=1}^{n} q_j \delta$$

$$= \delta$$

and

$$\left|p\left[p^{-1}\left(\sum_{j=1}^{n} q_j p(x_j)\right)\right] - f\left[p^{-1}\left(\sum_{j=1}^{n} q_j p(x_j)\right)\right]\right| < \delta.$$

Hence, $\rho([f],[p]) \leqslant \epsilon$.

We show now the existence of a polynomial P with rational coefficients that is strictly increasing on $[0,1]$ and such that $P(0) = 0, P(1) = 1$, and $|f(t) - P(t)| < \delta$ throughout $[0,1]$.

For each positive integer m, consider

$$B_m(t) \equiv \sum_{j=0}^{m} f\left(\frac{j}{m}\right)\binom{m}{j} t^j(1-t)^{m-j},$$

the Bernstein polynomial of order m of f. Clearly, $B_m(0) = 0$ and $B_m(1) = 1$. Moreover, $B_m'(t) > 0$ throughout $[0,1]$. In fact, a simple calculation (cf. [2], p. 12) shows that

$$B_m'(t) \equiv m \sum_{j=0}^{m-1} \left\{f\left(\frac{j+1}{m}\right) - f\left(\frac{j}{m}\right)\right\}\binom{m-1}{j} t^j(1-t)^{m-1-j}.$$

Since f is continuous on $[0,1]$, there exists a positive integer n ($\geqslant 2$) such that $|f(t) - B_n(t)| < \delta/2$ throughout $[0,1]$. Let $B_n(t) \equiv \sum_{k=0}^{n} b_k t^k$. If the b_k are all rational, we are finished. Suppose they are not all rational. Note that $0 = B_n(0) = b_0$ and that $1 = B_n(1) = b_1 + b_2 + \ldots + b_n$.

Let $\alpha = \min_{0 \leqslant t \leqslant 1} B_n'(t)$; note that $\alpha > 0$. Select rational numbers $a_2, a_3, \ldots, a_n$ such that $0 < a_k - b_k < \min((\delta/4n), (\alpha/n))$ $(k = 2,3,\ldots,n)$, and let $a_1 = 1 - \sum_{k=2}^{n} a_k$. Then

$$|b_1 - a_1| = \sum_{k=2}^{n} (a_k - b_k) < \frac{n-1}{4n}\delta.$$

Let $P(t) \equiv \sum_{k=1}^{n} a_k t^k$. Then P is a polynomial with rational coefficients, $P(0) = 0$, $P(1) = 1$, and throughout $[0,1]$, $|B_n(t) - P(t)| \leqslant \sum_{k=1}^{n} |b_k - a_k| t^k < (n-1)\delta(4n)^{-1} + (n-1)\delta(4n)^{-1} < \frac{1}{2}\delta$. Consequently, $\max_{0 \leqslant t \leqslant 1} |f(t) - P(t)| < \delta$.

To prove that P is strictly increasing on $[0,1]$, it is sufficient to show that $P'(t) > 0$ throughout $(0,1)$. But, if $0 < t < 1$, then $P'(t) - B_n'(t) > a_1 - b_1 = \sum_{k=2}^{n} (b_k - a_k) > -(n-1)\alpha/n > -\alpha$, and, therefore, $P'(t) > B_n'(t) - \alpha \geqslant 0$.

COROLLARY 2.1. *The metric space of equivalence classes of* CM[0,1] *is separable.*

3. THE METRIC SPACE OF CANONICAL FUNCTIONS

In this section we examine the metric space of equivalence classes in more detail and determine a number of its properties, by showing that it is homeomorphic to the metric space of canonical functions.

DEFINITION 3.1. If the real-valued function f is continuous on its domain $[0,1]$, then, for each $\delta \geqslant 0$, we denote $\omega_f(\delta) = \sup\{|f(x_1) - f(x_2)| : 0 \leqslant x_1 \leqslant 1,\ 0 \leqslant x_2 \leqslant 1,\ |x_1 - x_2| \leqslant \delta\}$.

DEFINITION 3.2. If h is a bounded, real-valued function with domain $[A, B]$, we denote $\|h\| = \sup\{|h(x)| : A \leqslant x \leqslant B\}$.

LEMMA 3.1. *If f and g are canonical functions, then*

$$\rho([f],[g]) \leqslant 2\omega_{f^{-1}}(\|f-g\|),$$

$$\|f^{-1} - g^{-1}\| \leqslant \rho([f],[g]),$$

and

$$\|f-g\| \leqslant \omega_f(\rho([f],[g])).$$

Proof. If $x = (x_1, x_2, \ldots, x_n)$ and $q = (q_1, q_2, \ldots, q_n)$ are admissible, then, as in the proof of Theorem 2.2,

$$\begin{aligned}|M_f(x,q) - M_g(x,q)| &\leqslant \left|f^{-1}\left\{\sum_{j=1}^{n} q_j f(x_j)\right\} - f^{-1}\left\{\sum_{j=1}^{n} q_j g(x_j)\right\}\right| \\ &\quad + \left|f^{-1}\left\{g\left[g^{-1}\left(\sum_{j=1}^{n} q_j g(x_j)\right)\right]\right\} - f^{-1}\left\{f\left[g^{-1}\left(\sum_{j=1}^{n} q_j g(x_j)\right)\right]\right\}\right| \\ &\leqslant 2\omega_{f^{-1}}(\|f-g\|),\end{aligned}$$

since

$$\left|\sum_{j=1}^{n} q_j f(x_j) - \sum_{j=1}^{n} q_j g(x_j)\right| \leqslant \sum_{j=1}^{n} q_j \|f-g\| = \|f-g\|$$

and

$$\left|g\left[g^{-1}\left(\sum_{j=1}^{n} q_j g(x_j)\right)\right] - f\left[g^{-1}\left(\sum_{j=1}^{n} q_j g(x_j)\right)\right]\right| \leqslant \|f-g\|.$$

This proves that $\rho([f],[g]) \leqslant 2\omega_{f^{-1}}(\|f-g\|)$.

Next, let $x_1 = 0$, $x_2 = 1$, $0 < y < 1$, $q_1 = 1 - y$, and $q_2 = y$. Then $M_f(x,q) = f^{-1}\{q_1 f(x_1) + q_2 f(x_2)\} = f^{-1}(y)$ and $M_g(x,q) = g^{-1}(y)$. Consequently, $|f^{-1}(y) - g^{-1}(y)| \leqslant \rho([f],[g])$ whenever $0 \leqslant y \leqslant 1$. This proves that $\|f^{-1} - g^{-1}\| \leqslant \rho([f],[g])$.

Finally, if $0 \leqslant x \leqslant 1$, then

$$|f(x) - g(x)| = |f\{g^{-1}(g(x))\} - f\{f^{-1}(g(x))\}| \leqslant \omega_f(\|g^{-1} - f^{-1}\|).$$

Hence, $\|f - g\| \leqslant \omega_f(\|f^{-1} - g^{-1}\|) \leqslant \omega_f(\rho([f],[g]))$.

THEOREM 3.1. *The metric space of equivalence classes of* CM[0,1] *is homeomorphic to the metric space of canonical functions with distance determined by the norm* $\| \ \|$.

Proof. Suppose that f is a canonical function. If $\epsilon > 0$, then there exists a $\delta > 0$ such that $2\omega_{f^{-1}}(\delta) < \epsilon$. If g is a canonical function such that $\|f - g\| < \delta$, then, by Lemma 3.1, $\rho([f],[g]) \leqslant 2\omega_{f^{-1}}(\|f - g\|) \leqslant 2\omega_{f^{-1}}(\delta) < \epsilon$. This proves that the transformation $f \to [f]$ is continuous at each point f.

Next, suppose that $[f]$ is an equivalence class of CM[0,1] where f is canonical. Given $\epsilon > 0$, there exists an $\eta > 0$ such that $\omega_f(\eta) < \epsilon$. If $[g]$ is an equivalence class of CM[0,1] such that g is canonical and $\rho([f],[g]) < \eta$, then by Lemma 3.1, $\|f - g\| \leqslant \omega_f(\rho([f],[g])) \leqslant \omega_f(\eta) < \epsilon$.

THEOREM 3.2. *The metric space of equivalence classes of* CM[0,1] *is arcwise connected and locally arcwise connected, but it is not compact.*

Proof. To prove that the space is arcwise connected, it will suffice to prove that the metric space of canonical functions is arcwise connected. Suppose that f and g are two distinct canonical functions, and let $f_\alpha = (1 - \alpha)f + \alpha g$ for every $\alpha \in [0,1]$. Then it is clear that each f_α is a canonical function. Moreover, $f_0 = f$ and $f_1 = g$. The mapping $\alpha \to f_\alpha$ is continuous at each point of $[0,1]$, is one-to-one, and, consequently, is a homeomorphism of $[0,1]$ onto $\{f_\alpha : 0 \leqslant \alpha \leqslant 1\}$, since $[0,1]$ is compact. Moreover, if $\|f - g\| < \epsilon$, then $\|f - f_\alpha\| = |\alpha| \cdot \|f - g\| < \alpha\epsilon \leqslant \epsilon$ whenever $0 < \alpha \leqslant 1$; this proves that the metric space of canonical functions is locally arcwise connected.

To prove that this space is not compact, consider the sequence of functions $f_1, f_2, f_3, \ldots$ where, for each positive integer n, $f_n(0) = 0$, $f_n(\frac{1}{2}) = 1 - 2^{-n}$, $f_n(1) = 1$, and f_n is linear on each of the intervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. Then each f_n is a canonical function. If the metric space of canonical functions were compact, then some subsequence $f_{n_1}, f_{n_2}, f_{n_3}, \ldots$ of the sequence $f_1, f_2, f_3, \ldots$ would converge to a canonical function f. *A fortiori*, $f_{n_1}(\frac{1}{2}), f_{n_2}(\frac{1}{2}), f_{n_3}(\frac{1}{2}), \ldots$ would converge to $f(\frac{1}{2})$, which is impossible, since $f_{n_k}(\frac{1}{2}) \to 1$ as $k \to \infty$ and $f(\frac{1}{2}) < f(1) = 1$.

THEOREM 3.3. *In the metric space of equivalence classes of* CM[0, 1], *no nonempty open set has a compact closure.*

Proof. Suppose that G is a nonempty open subset of the metric space of canonical functions whose closure $\bar{G}$ is compact. Let f belong to G; and let ϵ $(0 < \epsilon < 1)$ be such that $U(f,\epsilon) \subseteq G$, where $U(f,\epsilon)$ denotes the set of all canonical functions g for which $\|f - g\| < \epsilon$.

Let $\xi = \frac{1}{2}\{1 + f^{-1}(1-\epsilon)\}$; and consider the sequence of functions $g_1, g_2, g_3, \ldots$ where, for each positive integer n, $g_n(x) = f(x)$ if $0 \leqslant x \leqslant f^{-1}(1-\epsilon)$, $g_n(\xi) = 1 - \epsilon 2^{-n}$, $g_n(1) = 1$, and g_n is linear on each of the intervals $[f^{-1}(1-\epsilon), \xi]$ and $[\xi, 1]$. Then g_n belongs to $U(f,\epsilon)$ $(n = 1, 2, \ldots)$. Let $g_{n_1}, g_{n_2}, g_{n_3}, \ldots$ be a subsequence of $g_1, g_2, g_3, \ldots$ converging to a point g in $\bar{G}$. Thus, $\lim_{k\to\infty} g_{n_k}(\xi) = g(\xi) < 1 = \lim_{k\to\infty} g_{n_k}(\xi)$, a contradiction.

COROLLARY 3.1. *The metric space of equivalence classes of* CM[0, 1] *is not locally compact.*

COROLLARY 3.2. *In the metric space of equivalence classes of* CM[0, 1], *every compact set is nowhere dense.*

Proof. Let C be a compact subset of the space. The interior of C must be empty, by Theorem 3.3, since it is open and its closure is compact.

We note that the homeomorphism $f \to [f]$ is not isometric; in fact, it is not even uniformly continuous, as the following example shows. For each positive integer n, let f_n be the function with domain $[0, 1]$ such that $f_n(0) = 0$, $f_n(\frac{1}{2}) = 1 - 2^{-n}$, $f_n(1) = 1$, and such that f_n is linear on $[0, \frac{1}{2}]$ and on $[\frac{1}{2}, 1]$. If $m > n \geqslant 1$, then $\|f_m - f_n\| < 2^{-n}$; but, by Lemma 3.1, $\rho([f_m], [f_n]) \geqslant \|f_m^{-1} - f_n^{-1}\|$, and a simple calculation shows that $\|f_m^{-1} - f_n^{-1}\| \geqslant |f_m^{-1}(1 - 2^{-m}) - f_n^{-1}(1 - 2^{-m})| = \frac{1}{2}(1 - 2^{n-m}) \geqslant \frac{1}{4}$.

THEOREM 3.4. *The metric space of equivalence classes of* CM[0, 1] *is bounded and has diameter* 1, *but it is not totally bounded. In fact, if* $0 < \epsilon < 1$, *then the space contains no* ϵ*-net.*

Proof. That the diameter is 1 can be shown by using the second inequality of Lemma 3.1. Suppose that $0 < \epsilon < 1$. Then, given any finite set $\{[f_1], [f_2], \ldots, [f_n]\}$ of equivalence classes in CM[0, 1], we shall prove that there exists a function $f \in$ CM[0, 1] such that $\rho([f], [f_k]) \geqslant \epsilon$ for each $k = 1, 2, \ldots, n$. We assume, without loss of generality, that each f_k is canonical. For each $k = 1, 2, \ldots, n$, let δ_k be such that $0 < \delta_k < 1$ and $f_k^{-1}(x) < \frac{1}{2}(1 - \epsilon)$ whenever $0 \leqslant x \leqslant \delta_k$. Let $\delta = \min\{\delta_k : k = 1, 2, \ldots, n\}$. Let h be the function with domain

$[0,1]$ such that $h(0)=0$, $h(1)=1$, $h(\delta)=\epsilon+\frac{1}{2}(1-\epsilon)$, and such that h is linear on $[0,\delta]$ and on $[\delta,1]$. Let $f=h^{-1}$. Then, according to Lemma 3.1, for each $k=1,2,\ldots,n$,

$$\begin{aligned}\rho([f],[f_k]) &\geqslant \|f^{-1}-f_k^{-1}\| \\ &= \|h-f_k^{-1}\| \\ &\geqslant h(\delta)-f_k^{-1}(\delta) \\ &> \epsilon+\tfrac{1}{2}(1-\epsilon)-\tfrac{1}{2}(1-\epsilon) \\ &= \epsilon.\end{aligned}$$

4. Determination and Estimation of Distances

We now consider the question of determining explicitly the distance $\rho([f],[g])$ when the functions f and g are specified. We also consider the question of estimating the distance when the functions satisfy certain smoothness conditions.

For certain special functions f, g, the distance $\rho([f]),[g])$ can be determined at once from known complements of classical inequalities. We mention just one example. Shisha and Mond ([3], p. 301) have proved

Theorem 4.1. *Let $q_1,q_2,\ldots,q_n$ be positive numbers with $\sum_{j=1}^{n} q_j=1$, let $0<a<b$, $\gamma=b/a$, and let $a\leqslant x_j\leqslant b$ $(j=1,2,\ldots,n)$. Then*

$$\sum_{j=1}^{n} q_j x_j-\left(\sum_{j=1}^{n} q_j x_j^{-1}\right)^{-1}\leqslant (b^{1/2}-a^{1/2})^2.$$

Equality holds if and only if there exists a subsequence $(j_1,j_2,\ldots,j_p)$ of $(1,2,\ldots,n)$ such that $\sum_{m=1}^{p} q_{j_m}=(1+\gamma^{-1/2})^{-1}$, $x_{j_m}=b$ $(m=1,2,\ldots,p)$, and $x_j=a$ for every j distinct from all j_m.

This, in conjunction with the familiar fact that a (weighted) harmonic mean never exceeds the corresponding arithmetic mean, yields at once the following

Corollary 4.1. *Let $0<a<b$, and let $f(t)=t$, $g(t)=t^{-1}$, $a\leqslant t\leqslant b$. Then* $\rho([f],[g])=(b^{1/2}-a^{1/2})^2$.

Next, we estimate $\rho([f],[g])$ in several ways.

Theorem 4.2. *If f and g are canonical functions and if f^{-1} has a bounded derivative on* $[0,1]$, *then* $\rho([f],[g])\leqslant 2\|(f^{-1})'\|\cdot\|f-g\|$. (Here and below, first and second derivatives at end points mean onesided derivatives.)

Proof. According to Lemma 3.1, $\rho([f],[g]) \leqslant 2\omega_{f^{-1}}(\|f-g\|)$. The desired conclusion follows at once from the mean-value theorem.

THEOREM 4.3. *If $f, g \in \mathrm{CM}[a,b]$, then*

$$\rho([f],[g]) \leqslant \tfrac{1}{4}\|(f^{-1})'\|\cdot\|(f\circ g^{-1})''\|\cdot|g(b)-g(a)|^2,$$

provided the right-hand side of the inequality exists.

Proof. Let $h = f\circ g^{-1}$, let $n > 1$, let $x = (x_1, x_2, \ldots, x_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be admissible, and let $y_j = g(x_j)$ $(j = 1, 2, \ldots, n)$. By the mean-value theorem, for some α in the open interval joining $f(a)$ to $f(b)$, we have

$$\begin{aligned} M_f(x,q) - M_g(x,q) &= f^{-1}\left\{\sum_{j=1}^{n} q_j f(x_j)\right\} - f^{-1}\left[h\left\{\sum_{j=1}^{n} q_j g(x_j)\right\}\right] \\ &= (f^{-1})'(\alpha)\left[\sum_{j=1}^{n} q_j f(x_j) - h\left\{\sum_{j=1}^{n} q_j g(x_j)\right\}\right] \\ &= (f^{-1})'(\alpha)\left[\sum_{j=1}^{n} q_j h(y_j) - h\left\{\sum_{j=1}^{n} q_j y_j\right\}\right] \\ &= (f^{-1})'(\alpha)\left[\sum_{j=1}^{n} q_j\left\{h(y_j) - h\left(\sum_{k=1}^{n} q_k y_k\right)\right\}\right]. \end{aligned}$$

Using the mean-value theorem a second time, we conclude that there exist points $z_1, z_2, \ldots, z_n$ in the open interval joining $g(a)$ to $g(b)$, such that

$$\begin{aligned} M_f(x,q) - M_g(x,q) = (f^{-1})'(\alpha)\,[&q_1\{(1-q_1)y_1 - q_2 y_2 - \ldots - q_n y_n\}h'(z_1) \\ &+ q_2\{-q_1 y_1 + (1-q_2)y_2 - \ldots - q_n y_n\}h'(z_2) \\ &+ \ldots \\ &+ q_n\{-q_1 y_1 - q_2 y_2 - \ldots + (1-q_n)y_n\}h'(z_n)] \\ = (f^{-1})'(\alpha)\,[&q_1\{q_2(y_1-y_2) + \ldots + q_n(y_1-y_n)\}h'(z_1) \\ &+ q_2\{q_1(y_2-y_1) + \ldots + q_n(y_2-y_n)\}h'(z_2) \\ &+ \ldots \\ &+ q_n\{q_1(y_n-y_1) + \ldots + q_{n-1}(y_n-y_{n-1})\}h'(z_n)] \\ = (f^{-1})'(\alpha) &\sum_{1\leqslant i<j\leqslant n} q_i q_j (y_i - y_j)\{h'(z_i) - h'(z_j)\}. \end{aligned}$$

Using the mean-value theorem a third time, we conclude that there exist points w_{ij} $(1 \leqslant i < j \leqslant n)$ in the open interval joining $g(a)$ to $g(b)$, such that

$$(f^{-1})'(\alpha) \sum_{1 \leqslant i < j \leqslant n} q_i q_j (y_i - y_j)\{h'(z_i) - h'(z_j)\}$$
$$= (f^{-1})'(\alpha) \sum_{1 \leqslant i < j \leqslant n} q_i q_j (y_i - y_j)(z_i - z_j) h''(w_{ij}).$$

Consequently,

$$|M_f(x,q) - M_g(x,q)| \leqslant |(f^{-1})'(\alpha)| \sum_{1 \leqslant i < j \leqslant n} q_i q_j |y_i - y_j| \cdot |z_i - z_j| \cdot |h''(w_{ij})|$$
$$\leqslant \|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(b) - g(a)|^2 \sum_{1 \leqslant i < j \leqslant n} q_i q_j.$$

Next, let us prove that $\sum_{1 \leqslant i < j \leqslant n} q_i q_j < \frac{1}{2}$. By Cauchy's inequality,

$$1 = \left(\sum_{j=1}^{n} 1 \cdot q_j\right)^2 \leqslant n \sum_{j=1}^{n} q_j^2.$$

Thus,

$$\sum_{j=1}^{n} q_j^2 \geqslant \frac{1}{n}.$$

Therefore,

$$\sum_{1 \leqslant i < j \leqslant n} q_i q_j = \frac{1}{2}\left[\left(\sum_{j=1}^{n} q_j\right)^2 - \sum_{j=1}^{n} q_j^2\right]$$
$$= \frac{1}{2}\left[1 - \sum_{j=1}^{n} q_j^2\right]$$
$$\leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

This establishes Theorem 4.3 if the multiplicative factor $\frac{1}{4}$ is replaced by $\frac{1}{2}$.

To show that the multiplicative factor can be taken to be $\frac{1}{4}$, we prove first

LEMMA 4.1. *Suppose that* $n > 1$, $A < B$, $q_j > 0$ $(j = 1,2,\ldots,n)$, $\sum_{j=1}^{n} q_j = 1$, *and* $A \leqslant Y_j \leqslant B$ $(j = 1,2,\ldots,n)$. *Then*

$$\sum_{1 \leqslant i < j \leqslant n} q_i q_j (Y_i - Y_j) \leqslant \tfrac{1}{4}(B - A).$$

Equality holds if and only if there exists an integer J *such that* $1 \leqslant J < n$, $q_1 + q_2 + \ldots + q_J = q_{J+1} + \ldots + q_n$, $Y_1 = Y_2 = \ldots = Y_J = B$, *and* $Y_{J+1} = \ldots = Y_n = A$.

Proof. $\sum_{1\leqslant i<j\leqslant n} q_i q_j (Y_i - Y_j) = \sum_{j=1}^{n} \alpha_j Y_j$ where

$$\alpha_1 = q_1(q_2 + q_3 + q_4 + \ldots + q_n),$$
$$\alpha_2 = q_2(-q_1 + q_3 + q_4 + \ldots + q_n),$$
$$\alpha_3 = q_3(-q_1 - q_2 + q_4 + \ldots + q_n),$$
$$\vdots$$
$$\alpha_n = q_n(-q_1 - q_2 - q_3 - \ldots - q_{n-1}).$$

Note that $\alpha_1 > 0$, $\alpha_n < 0$. Moreover, if $\alpha_j \leqslant 0$ for some $j < n$, then $\alpha_{j+1} < 0$, since

$$\frac{\alpha_{j+1}}{q_{j+1}} = \frac{\alpha_j}{q_j} - q_{j+1} - q_j.$$

This proves that there exists an integer J, $1 \leqslant J < n$, such that $\alpha_j \geqslant 0$ if $j \leqslant J$, and $\alpha_j < 0$ if $j > J$. Let $Y_j' = B$ if $j \leqslant J$, $Y_j' = A$ if $j > J$. Then, clearly,

$$\begin{aligned}
\sum_{1\leqslant i<j\leqslant n} q_i q_j (Y_i - Y_j) &= \sum_{j=1}^{n} \alpha_j Y_j \\
&\leqslant \sum_{j=1}^{n} \alpha_j Y_j' \\
&= \sum_{1\leqslant i<j\leqslant n} q_i q_j (Y_i' - Y_j') \\
&= \sum_{i=1}^{J} \sum_{j=J+1}^{n} q_i q_j (B - A) \\
&= \left(\sum_{i=1}^{J} q_i\right)\left(\sum_{j=J+1}^{n} q_j\right)(B - A) \\
&\leqslant \tfrac{1}{4}(B - A),
\end{aligned}$$

as desired. If equality holds, $q_1 + q_2 + \ldots + q_J = q_{J+1} + \ldots + q_n = \frac{1}{2}$. Then $\alpha_J = q_J{}^2 > 0$; and, hence, $Y_1 = Y_2 = \ldots = Y_J = B$ and $Y_{J+1} = \ldots Y_n = A$. The sufficiency of the condition for equality also follows easily.

Returning to the proof of Theorem 4.3, let $(k_1, k_2, \ldots, k_n)$ be a permutation of $(1, 2, \ldots, n)$ such that $y_{k_1} \geqslant y_{k_2} \ldots \geqslant y_{k_n}$. Then

$$\begin{aligned}
&|M_f(x,q) - M_g(x,q)| \\
&\qquad \leqslant \|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(b) - g(a)| \sum_{1\leqslant i<j\leqslant n} q_{k_i} q_{k_j} (y_{k_i} - y_{k_j}) \\
&\qquad \leqslant \tfrac{1}{4} \|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(b) - g(a)|^2.
\end{aligned}$$

COROLLARY 4.2. *If $f, g \in \mathrm{CM}[a,b]$, then*

$$\rho([f],[g]) \leqslant \frac{1}{4}\left\|\frac{1}{f'}\right\|\cdot\left\|\frac{1}{g'}\left(\frac{f'}{g'}\right)'\right\|\cdot|g(b)-g(a)|^2,$$

provided the right-hand side of the inequality exists.

Proof. This follows at once from the facts that

$$(f^{-1})' = \frac{1}{f' \circ f^{-1}}$$

and

$$(f\circ g^{-1})'' = \frac{(g'\circ g^{-1})(f''\circ g^{-1}) - (f'\circ g^{-1})(g''\circ g^{-1})}{(g'\circ g^{-1})^3}$$
$$= \left[\frac{1}{g'}\left(\frac{f'}{g'}\right)'\right]\circ g^{-1}.$$

COROLLARY 4.3. *If $f, g \in \mathrm{CM}[a,b]$, then*

$$\rho([f]),[g]) \leqslant \tfrac{1}{2}\|(f^{-1})'\|\cdot\|(f\circ g^{-1})'\|\cdot|g(b)-g(a)|$$
$$= \frac{1}{2}\left\|\frac{1}{f'}\right\|\cdot\left\|\frac{f'}{g'}\right\|\cdot|g(b)-g(a)|,$$

provided that all expressions involved exist.

Proof. In the proof of Theorem 4.3, note that $|h'(z_i) - h'(z_j)| \leqslant 2\|h'\|$. Then use the fact that $(f\circ g^{-1})' = (f'\circ g^{-1})/(g'\circ g^{-1})$.

Remarks. (1) In connection with Theorem 4.3, we note that, if $F \sim f$ and $G \sim g$, then, as one would expect, and as a straightforward calculation shows,

$$\|(F^{-1})'\|\cdot\|(F\circ G^{-1})''\|\cdot|G(b)-G(a)|^2$$
$$= \|(f^{-1})'\|\cdot\|(f\circ g^{-1})''\|\cdot|g(b)-g(a)|^2.$$

(2) If $f(t) = t$ and $g(t) = t^{1/2}$ for each $t \in [0,1]$, then

$$2\|(f^{-1})'\|\cdot\|f-g\| = \tfrac{1}{2} = \tfrac{1}{4}\|(f^{-1})'\|\cdot\|(f\circ g^{-1})''\|\cdot|g(1)-g(0)|^2.$$

In this case, Theorems 4.2 and 4.3 give the same estimate. From Lemma 3.1 we conclude that $\rho([f],[g]) \geqslant \|f^{-1} - g^{-1}\| = \tfrac{1}{4}$.

(3) Let $h(t) = t\sin(\pi t/2)$ for each $t \in [0,1]$. Then $h(0) = 0$, $h(1) = 1$, and

$$h'(t) = t\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right) > 0$$

if $0 < t \leqslant 1$. Thus, h is a canonical function, and so is its inverse. Let $f(t) = t$ and $g(t) = h^{-1}(t)$ throughout $[0,1]$. Then $\|(f^{-1})'\| = 1$, and

$$(f \circ g^{-1})''(t) \equiv h''(t)$$
$$\equiv -t\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}t\right) + \pi \cos\left(\frac{\pi}{2}t\right).$$

Since

$$h'''(t) = -t\left(\frac{\pi}{2}\right)^3 \cos\left(\frac{\pi}{2}t\right) - \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}t\right) - \frac{\pi^2}{2}\sin\left(\frac{\pi}{2}t\right) < 0$$

if $0 < t \leqslant 1$, it follows that $\|(f \circ g^{-1})''\| = \pi$. Hence,

$$\tfrac{1}{4}\|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(1) - g(0)|^2 = \frac{\pi}{4}.$$

In this case, $2\|(f^{-1})'\| \cdot \|f - g\|$ is not easy to evaluate. Thus, Theorem 4.3 is sometimes easier to use than Theorem 4.2.

(4) If $f(t) = t$ and $h(t) = \frac{1}{3}(t^3 + 2t)$ for each $t \in [0,1]$, and if $g = h^{-1}$, then $\frac{1}{4}\|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(1) - g(0)|^2 = \frac{1}{2}$. Next, let us prove that $\|f - g\| = \frac{2}{9}\sqrt{\frac{1}{3}}$. Clearly, $g^{-1}(t) = \frac{1}{3}(t^3 + 2t) \leqslant t$ if $0 \leqslant t \leqslant 1$. Consequently, $f(t) = t \leqslant g(t)$ throughout $[0,1]$. Moreover, $g(0) - f(0) = g(1) - f(1) = 0$, and $(d/dt)[g(t) - f(t)] = 0$ if and only if $g'(t) = 1$. But $\frac{1}{3}[g^3(t) + 2g(t)] \equiv t$, and so $\frac{1}{3}[3g^2(t)g'(t) + 2g'(t)] \equiv 1$. If $0 < t < 1$ and $g'(t) = 1$, then $\frac{1}{3}[3g^2(t) + 2] = 1$, $g(t) = \sqrt{\frac{1}{3}}$, and $t = \frac{1}{3}[(\sqrt{\frac{1}{3}})^3 + 2\sqrt{\frac{1}{3}}] = \frac{7}{9}\sqrt{\frac{1}{3}}$. Consequently, $\|f - g\|$ must be equal to $g(\frac{7}{9}\sqrt{\frac{1}{3}}) - f(\frac{7}{9}\sqrt{\frac{1}{3}}) = \sqrt{\frac{1}{3}} - \frac{7}{9}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{\frac{1}{3}}$. Hence,

$$\tfrac{1}{4}\|(f^{-1})'\| \cdot \|(f \circ g^{-1})''\| \cdot |g(1) - g(0)|^2 = \tfrac{1}{2}$$
$$> \tfrac{4}{9}\sqrt{\tfrac{1}{3}}$$
$$= 2\|(f^{-1})'\| \cdot \|f - g\|;$$

and, in this case, Theorem 4.2 gives a sharper estimate then Theorem 4.3.

5. Some Further Remarks

Let $C[0,1]$ denote, as usual, the metric space of all continuous, real-valued functions with domain $[0,1]$. Since every subset of a separable metric space is separable, it follows that the metric space of all canonical functions is, like $C[0,1]$, separable. Hence, Corollary 2.1 is a consequence of Theorem 3.1; and, in a sense, Theorem 2.2 is redundant. However, our motivation for including Theorem 2.2 was that it raises a number of interesting questions belonging to approximation theory.

Also, a natural question to ask is, whether or not the metric space of equivalence classes of CM[0, 1] is complete. If it is not, how does its completion compare with that of the metric space of canonical functions?

6. Simultaneous Reduction of Generalized Means

In the remark following Definition 2.3, we asserted that the supremum in that definition is actually attained. This assertion is an immediate consequence of the following Theorem 6.1, since the function

$$|f^{-1}\{Q_1 f(X_1) + Q_2 f(X_2)\} - g^{-1}\{Q_1 g(X_1) + Q_2 g(X_2)\}|$$

is continuous on the compact subset

$$\{(X_1, X_2, Q_1, Q_2): a \leqslant X_1 \leqslant b, a \leqslant X_2 \leqslant b, Q_1 \geqslant 0, Q_2 \geqslant 0, Q_1 + Q_2 = 1\}$$

of Euclidean 4-space.

Theorem 6.1. *Suppose that* $f,g \in \mathrm{CM}[a,b]$. *If* $x = (x_1, x_2, \ldots, x_n)$ *and* $q = (q_1, q_2, \ldots, q_n)$ *are admissible, then there exist admissible* $X = (X_1, X_2)$ *and* $Q = (Q_1, Q_2)$ *such that* $M_f(x,q) = M_f(X, Q)$ *and* $M_g(x,q) = M_g(X, Q)$.

Proof. Note that

$$A = \left(\sum_{j=1}^{n} q_j f(x_j), \sum_{j=1}^{n} q_j g(x_j)\right) = \sum_{j=1}^{n} q_j (f(x_j), g(x_j))$$

is a point in the convex hull of the curve $\Gamma = \{(f(x), g(x)): a \leqslant x \leqslant b\}$. Also, note that Γ is a connected subset of the plane. If A lies on Γ, then it is equal to $(f(X), g(X))$ for some $X \in [a,b]$. In this case, let $X_1 = X_2 = X$ and $Q_1 = Q_2 = \frac{1}{2}$.

If $A \notin \Gamma$, then, according to an extension of Carathéodory's theorem (cf. [4], p. 35), there exist two distinct points in Γ, say, $(f(X_1), g(X_1))$ and $(f(X_2), g(X_2))$, where $X_1, X_2 \in [a,b]$, such that the line segment joining them contains A. Thus, there exist positive numbers Q_1, Q_2, with $Q_1 + Q_2 = 1$, such that

$$Q_1 f(X_1) + Q_2 f(X_2) = \sum_{j=1}^{n} q_j f(x_j)$$

and

$$Q_1 g(X_1) + Q_2 g(X_2) = \sum_{j=1}^{n} q_j g(x_j).$$

This completes the proof of the theorem.

Remarks. (1) Theorem 6.1 can obviously be generalized.

(2) It is not possible to strengthen the conclusion of Theorem 6.1 by asserting that X_1 can always be taken to be either a or b. To see this, let $f(x) = x$ throughout $[0, 1]$, let g be the function whose graph is shown in Figure 1, and consider the point P, which is in the convex hull of the curve $\{(f(x), g(x)) : 0 \leqslant x \leqslant 1\}$.

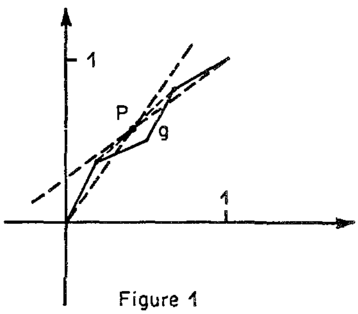

Figure 1

(3) The proof of Theorem 4.3 can be substantially simplified by using Theorem 6.1.

(4) The following is a simple application of Theorem 6.1. If $0 < a \leqslant X_1 \leqslant b$, $a \leqslant X_2 \leqslant b$, $Q_1 > 0$, $Q_2 > 0$, and $Q_1 + Q_2 = 1$, then

$$\begin{aligned}(Q_1 X_1 + Q_2 X_2)(Q_1 X_1^{-1} + Q_2 X_2^{-1}) &= Q_1{}^2 + Q_2{}^2 + Q_1 Q_2\left(\frac{X_1}{X_2} + \frac{X_2}{X_1}\right)\\ &= (Q_1 + Q_2)^2 + Q_1 Q_2\left(\frac{X_1}{X_2} + \frac{X_2}{X_1} - 2\right)\\ &= 1 + Q_1 Q_2\left(\frac{X_1}{X_2} + \frac{X_2}{X_1} - 2\right)\\ &\leqslant 1 + \frac{1}{4}\left(\frac{a}{b} + \frac{b}{a} - 2\right) = \frac{(a+b)^2}{4ab}.\end{aligned}$$

It now follows from Theorem 6.1 that, if $0 < a \leqslant x_j \leqslant b$ $(j = 1, 2, \ldots, n)$, $q_j > 0$ $(j = 1, 2, \ldots, n)$, and $\sum_{j=1}^{n} q_j = 1$, then

$$\left(\sum_{k=1}^{n} q_k x_k\right)\left(\sum_{k=1}^{n} \frac{q_k}{x_k}\right) \leqslant \frac{(a+b)^2}{4ab},$$

a well-known inequality due to Kantorovich.

REFERENCES

1. G. H. HARDY, J. E. LITTLEWOOD AND G. PÓLYA, "Inequalities," 2nd ed. Cambridge Univ. Press, London and New York, 1952.
2. G. G. LORENTZ, "Bernstein Polynomials." University of Toronto Press, Toronto, 1953.

3. O. Shisha and B. Mond, Bounds on differences of means, in "Inequalities" (Proceedings of a Symposium, O. Shisha, ed.). Academic Press, New York and London, 1967.
4. H. G. Eggleston, "Convexity." Cambridge Univ. Press, London and New York, 1958.
5. O. Shisha and G. T. Cargo, On comparable means. *Pacific J. Math.* **14** (1964), 1053–1058.
6. G. T. Cargo and O. Shisha, Bounds on ratios of means. *J. Res. Nat. Bur. Standards* **66B** (1962), 169–170.